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Foldy–Wouthuysen transformation for a spinning particle with anomalous magnetic moment

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Abstract
In this paper we study the Foldy–Wouthuysen transformation for a pseudoclassical particle with anomalous magnetic moment in an external classical stationary electromagnetic field. We show that the transformation can be expressed in a closed form for neutral particles in purely electrostatic fields and for neutral and charged particles in external magnetostatic fields. The explicit expressions of the diagonalized Hamiltonians are calculated.

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1. Introduction

In past years Grassmann variables proved a very useful instrument to describe pseudoclassical spinning relativistic particles and superparticles and to investigate the properties of their interactions. Many applications were made to electromagnetic couplings [1–7], and to interactions of more general nature as, for instance, gauge [8–11] and gravitational fields [12–14]. All these models were first connected with properties emerging from string theory and later on became interesting in themselves. Their quantum structure was thoroughly investigated and, in the framework of electromagnetic interactions, the Foldy–Wouthuysen transformation (hereafter FWT) in the presence of a stationary magnetic field was determined [15]. The method to reach this result was the usual one, based on a graded canonical transformation [16] which reduces the wave equation to a representation where the Hamiltonian is an even matrix, in the typical form of a square-root operator containing both kinetic and interaction energies. More refined results were then obtained when the external fields were taken in the form of plane waves. In this case, in analogy with well-established results [17], it was proven by path integral [18–22] and by canonical theory [23, 24] that the semi-classical approximation reproduced the exact quantum propagator. The description of spinning particles was finally generalized by allowing for the presence of anomalous magnetic moment, first introduced in [25] and subsequently in [26–28], all these treatments leading to the same first class Dirac constraints and hence to the same wave equation. More recently, by using this approach, we considered the quantization of a spinning particle with anomalous magnetic moment in the field of an electromagnetic plane wave [24], generalizing the results obtained in [22]. We found that the semi-classical approximation is no longer exact, but for some particular cases, due to the effects arising from the interference of the anomalous magnetic moment with the electric charge that requires the presence of a T-ordered product in the quantum propagator.

The purpose of this work is to extend the FWT to a pseudoclassical spinning particle with anomalous magnetic moment in a classical stationary electromagnetic field, thus completing the research begun in [15] for the usual spinning particle. We are particularly interested in studying the cases in which the result can be expressed in a closed form. These turn out to be the following: (a) a neutral particle in a stationary electric field; (b) a neutral particle in a stationary magnetic field; (c) a charged particle in a stationary magnetic field. Observe that all the external electromagnetic fields we deal with, as well as the gauge and gravitational fields that appear in all the quoted papers, are strictly classical and the description of the matter is done in a one-particle first-quantized framework [29, 30]. The technique that yields the results is not so different from the one we used, for instance, in [24]. A crucial point that distinguishes this paper from our previous ones, however, is the need to exploit two possible different ways of realizing the quantization of the Dirac pseudoclassical brackets: in fact these representations of the Clifford algebra, coming from the quantization of the pseudoclassical variables, give rise to two different expressions for the Dirac equation, intertwined by a Pauli–Gursey unitary transformation [19, 31], that allow us to get more leisurely the exact form for the three different interacting cases quoted above.

The content of this paper can be summarized as follows. In section 2 we briefly recall the quantization scheme of the pseudoclassical particle with anomalous magnetic moment: we write the singular Lagrangian, the Dirac constraints and their corresponding operator form leading to the wave equation. We then formulate the FWT problem and we consider its general features. In section 3 we present a detailed discussion of the results concerning the three cases in which the transformation can be expressed in a closed form.

2. The general setting of the FWT for the spinning particle with anomalous magnetic moment

For the sake of completeness in this section we briefly summarize our notations and we report the Dirac constraints leading to the canonical quantization of the pseudoclassical particle with anomalous magnetic moment. The details can be found in [24, 25]. With the usual conventions for the metric tensor and for the gamma matrices [29], in a unit system with $\hbar = c = 1$, the Lagrangian we start with is [25]

$$\begin{aligned}\mathcal{L}(x_\mu, \dot{x}_\mu, \xi_\mu, \dot{\xi}_\mu, \xi_5, \dot{\xi}_5) &= -\frac{\mathrm{i}}{2}(\xi \cdot \dot{\xi}) - \frac{\mathrm{i}}{2}\xi_5\dot{\xi}_5 - q(\dot{x} \cdot A)\\ &\quad - \left[m^2 - \mathrm{i}\left(q + \frac{e\mu}{2}\right)F_{\mu\nu}\xi^\mu\xi^\nu - \frac{\mathrm{e}^2\mu^2}{16m^2}F_{\mu\nu}F_{\rho\sigma}\xi^\mu\xi^\nu\xi^\rho\xi^\sigma\right]^{1/2}\\ &\quad \times\left[\left(\dot{x}^\mu - \mathrm{i}\left(m + \frac{\mathrm{i}e\mu}{4m}F_{\lambda\nu}\xi^\lambda\xi^\nu\right)^{-1}\xi^\mu\left(\dot{\xi}_5 - \frac{e\mu}{2m}\dot{x}^\rho F_{\rho\sigma}\xi^\sigma\right)\right)^2\right]^{1/2}. \end{aligned} \tag{2.1}$$

Here x^μ are the usual space-time coordinates and (ξ^μ, ξ_5) are Grassmann variables related to the spin structure of the pseudoclassical particle. Moreover $\mu = -\Delta g = -(g - 2)$ where g

is the gyromagnetic factor [29, 30, 32], q is the charge of the particle and e is the electronic charge, respectively.

The Lagrangian (2.1) is evidently singular and gives rise to the two first class constraints

$$\chi_D = (\Pi \cdot \xi) - m\xi_5 + \mathrm{i}\frac{e\mu}{4m} F_{\mu\nu}\xi^\mu\xi^\nu\xi_5,$$

$$\chi = \Pi^2 - m^2 + \mathrm{i}\left(q + \frac{e\mu}{2}\right) F_{\mu\nu}\xi^\mu\xi^\nu + \mathrm{i}\frac{e\mu}{m}\Pi^\mu F_{\mu\nu}\xi^\nu\xi_5 + \frac{\mathrm{e}^2\mu^2}{16m^2} F_{\mu\nu}F_{\rho\sigma}\xi^\mu\xi^\nu\xi^\rho\xi^\sigma,$$

where the kinetic momentum Π is related to the canonical momentum p by

$$\Pi^\mu = p^\mu - qA^\mu \tag{2.2}$$

and the second class constraints have already been accounted for. Their algebra

$$\{\chi_D, \chi_D\} = \mathrm{i}\chi, \qquad \{\chi_D, \chi\} = 0, \qquad \{\chi, \chi\} = 0$$

is determined by the nonvanishing Dirac brackets of the pseudoclassical variables,

$$\{x^\mu, p^\nu\} = -\eta^{\mu\nu} \qquad \{\xi^\mu, \xi^\nu\} = \mathrm{i}\eta^{\mu\nu}, \qquad \{\xi_5, \xi_5\} = -\mathrm{i},$$

that, upon quantization, give rise to the graded commutators

$$[x^\mu, p^\nu] = -\mathrm{i}\eta^{\mu\nu}, \qquad \{\hat{\xi}^\mu, \hat{\xi}^\nu\}_+ = -\eta^{\mu\nu}, \qquad \{\hat{\xi}_5, \hat{\xi}_5\}_+ = 1. \tag{2.3}$$

It was previously observed [2, 19, 22] that the anti-commutation relations (2.3) of the odd operators $\hat{\xi}^\mu$, $\hat{\xi}_5$, can be satisfied by two different realizations

$$\hat{\xi}^\mu = 2^{-1/2}\gamma_5\gamma^\mu, \qquad \hat{\xi}_5 = 2^{-1/2}\gamma_5 \tag{2.4}$$

$$\hat{\xi}^\mu = 2^{-1/2}\mathrm{i}\gamma^\mu, \qquad \hat{\xi}_5 = 2^{-1/2}\gamma_5 \tag{2.5}$$

hereby referred to as (D) and (PG), respectively. It was also observed, [19], that these two realizations are connected by a Pauli–Gursey transformation, i.e. a conjugation by the matrix $\exp[\mathrm{i}(\pi/4)\gamma_5]$. However, contrary to almost all the previously quoted papers, where only the realization (D) was effectively used, in the following both (D) and (PG) will appear, since the different cases we will examine are treated more efficiently if the appropriate choice is made. An observation is however in order. When applying the Pauli–Gursey transformation to the Dirac equation, both the Hamiltonian and the spinor wavefunctions are transformed. Since we will use the standard representation of the γ-matrices where $\gamma^0 = \beta$ is diagonal and γ_5 is not, [29], the components of the original spinor will mix, so that some care must be used when performing the non-relativistic limit on the transformed Hamiltonian: the real advantage of going to the (PG) realization is actually apparent mainly when the FWT can be given a closed form. We will give further comments for each case we examine in the following.

The explicit form of the quantized Dirac Hamiltonian in the (D)realization (2.4) takes the form

$$\hat{H}_{\mathrm{D}} = (\vec{\alpha} \cdot \vec{\Pi}) + qA_0 + \beta m + \frac{e\mu}{8m}\beta\sigma_{\mu\nu}F^{\mu\nu}, \tag{2.6}$$

where $\beta = \gamma^0$, $\vec{\alpha} = \gamma^0\vec{\gamma}$ and $\sigma_{\mu\nu} = (\mathrm{i}/2)[\gamma_\mu, \gamma_\nu]$, [29]. In the (PG) realization (2.5) we have, instead,

$$\hat{H}_{\mathrm{PG}} = (\vec{\alpha} \cdot \vec{\Pi}) + qA_0 - \mathrm{i}\beta\gamma_5 m - \frac{\mathrm{i}e\mu}{8m}\beta\gamma_5\sigma_{\mu\nu}F^{\mu\nu} \tag{2.7}$$

and we can easily verify that

$$\mathrm{e}^{\mathrm{i}\pi\gamma_5/4}\hat{H}_{\mathrm{D}}\,\mathrm{e}^{-\mathrm{i}\pi\gamma_5/4} = \hat{H}_{\mathrm{PG}}.$$

In view of the discussion of the FWT, we find it useful to separate, both in $\hat{H}_{\mathrm{D}}$ and in $\hat{H}_{\mathrm{PG}}$, the even and the odd terms. We therefore write

$$\hat{H}_{\mathrm{D}} = \hat{H}_{\mathrm{D}}^{\mathrm{even}} + \hat{H}_{\mathrm{D}}^{\mathrm{odd}}, \qquad \hat{H}_{\mathrm{PG}} = \hat{H}_{\mathrm{PG}}^{\mathrm{even}} + \hat{H}_{\mathrm{PG}}^{\mathrm{odd}}, \tag{2.8}$$

and making explicit the electromagnetic tensor $F^{\mu\nu}$ in (2.6) and (2.7) we obtain

$$\hat{H}_{\mathrm{D}}^{\mathrm{even}} = qA_0 + \beta m - \frac{e\mu}{4m}\beta(\vec{\Sigma}\cdot\vec{B}) \qquad \hat{H}_{\mathrm{D}}^{\mathrm{odd}} = (\vec{\alpha}\cdot\vec{\Pi}) + \frac{\mathrm{i}e\mu}{4m}\beta(\vec{\alpha}\cdot\vec{E}) \tag{2.9}$$

$$\hat{H}_{\mathrm{PG}}^{\mathrm{even}} = qA_0 + \frac{e\mu}{4m}\beta\gamma_5(\vec{\alpha}\cdot\vec{E}) \qquad \hat{H}_{\mathrm{PG}}^{\mathrm{odd}} = (\vec{\alpha}\cdot\vec{\Pi}) - \mathrm{i}m\beta\gamma_5 + \frac{\mathrm{i}e\mu}{4m}\beta\gamma_5(\vec{\Sigma}\cdot\vec{B}), \tag{2.10}$$

where the spatial spin vector $\vec{\Sigma}$, defined by the relation $\sigma^{ij} = \epsilon^{ijk}\Sigma^k$, can also be written as $\vec{\Sigma} = \gamma_5\vec{\alpha}$.

It is well known that in the simplest systems, as for example the Dirac-free particle and the Dirac particle with no anomalous magnetic moment in a magnetic field, the FWT depends upon the odd part of the Hamiltonian that, in the two mentioned cases, is given by the kinetic part $(\vec{\alpha}\cdot\vec{\Pi})$ and anti-commutes with the even part βm. We will follow a similar method also for the more general interacting case with the additional difficulties we will discuss later on. We therefore present an extremely rapid summary of the successive steps necessary to get the result in these two simplest cases and we give the explicit forms of the operator $\exp[\beta\mathcal{O}]$ that defines the FWT, $\mathcal{O}$ being the odd part of the Hamiltonian operator [29, 33]. Starting from the non-interacting case, obtained by (2.6) and (2.7) with $\mu = q = 0$, it can be seen that for the first realization (2.4) of the Clifford algebra, the unitary transformation is generated by

$$\exp[\mathrm{i}\hat{S}_{\mathrm{D}}] = \exp[\beta(\vec{\alpha}\cdot\vec{p})\theta(|\vec{p}|)] \qquad \text{where} \quad \theta(|\vec{p}|) = \frac{1}{2|\vec{p}|}\arctan\frac{|\vec{p}|}{m}. \tag{2.11}$$

The FW transformed free-Hamiltonian operator is then [29]

$$\widetilde{\hat{H}}_{\mathrm{D}} = \mathrm{e}^{\beta(\vec{\alpha}\cdot\vec{p})\theta(|\vec{p}|)}((\vec{\alpha}\cdot\vec{p}) + \beta m)\,\mathrm{e}^{-\beta(\vec{\alpha}\cdot\vec{p})\theta(|\vec{p}|)} = \beta[\vec{p}^{\,2} + m^2]^{1/2}. \tag{2.12}$$

For the realization (2.5) of the Clifford algebra the whole free-Hamiltonian $\hat{H}_{\mathrm{PG}}$ is odd, as the mass term $-\mathrm{i}m\beta\gamma_5$ is itself odd (and still anti-commuting with $(\vec{\alpha}\cdot\vec{\Pi})$). The expression (2.11) has therefore to be substituted by

$$\exp[\mathrm{i}\hat{S}_{\mathrm{PG}}] = \exp[\beta((\vec{\alpha}\cdot\vec{p}) - \mathrm{i}m\beta\gamma_5)\phi(|\vec{p}|)] \qquad \text{where} \quad \phi(|\vec{p}|) = \frac{\pi}{4[\vec{p}^{\,2} + m^2]^{1/2}}.$$

As expected, the transformed free-Hamiltonian reads again $\widetilde{\hat{H}}_{\mathrm{PG}} = \beta[\vec{p}^{\,2} + m^2]^{1/2}$.

The Hamiltonian operators for the pseudoclassical particle interacting with a stationary magnetic field in the two representations are obtained from (2.9) and (2.10) by choosing $\mu = 0$, $A_0 = 0$ and $\vec{A} = \vec{A}(\vec{x})$. The computations are somewhat more cumbersome, but can still be managed and give a result in a closed form. We introduce

$$\hat{\Lambda} = -\{(\hat{\vec{\xi}}\cdot\vec{\Pi}), (\hat{\vec{\xi}}\cdot\vec{\Pi})\}_+ = \frac{1}{2}\{(\vec{\gamma}\cdot\vec{\Pi}), (\vec{\gamma}\cdot\vec{\Pi})\}_+ = -\left(\vec{\Pi}^2 + \frac{q}{2}\sigma^{ij}F^{ij}\right)$$

and by quantizing the graded Jacobi identity

$$\sum_{\text{cyclic}} (-1)^{d_\ell d_n}\{v_\ell, \{v_m, v_n\}\} = 0,$$

where υ_i is a generic dynamical variable of degree $d_i = 0, 1$ according to its parity in the Grassmann algebra, we easily verify that $[\beta(\vec{\alpha}\cdot\vec{\Pi}), \hat{\Lambda}] = 0$. By a direct calculation it can then be proved that the similarity transformation of $\hat{H}_{\mathrm{D}}$ with

$$\exp[\mathrm{i}\hat{S}_{\mathrm{D}}] = \exp[\beta(\vec{\alpha}\cdot\vec{\Pi})\theta(\hat{\Lambda})], \qquad \text{where} \quad \theta(\hat{\Lambda}) = \frac{1}{2\sqrt{\hat{\Lambda}}}\arctan\frac{\sqrt{\hat{\Lambda}}}{m} \tag{2.13}$$

gives the Hamiltonian [15]

$$\widetilde{\hat{H}}_{\mathrm{D}} = \beta[\vec{\Pi}^2 - q(\vec{\Sigma}\cdot\vec{B}) + m^2]^{1/2}. \tag{2.14}$$

Note that (2.14) is the same expression obtained for a stationary and uniform magnetic field, as found in [34]. The previous result can also be deduced by transforming (2.10), where now the complete Hamiltonian is odd, by

$$\exp[\mathrm{i}\hat{S}_{\mathrm{PG}}] = \exp[\beta((\vec{\alpha}\cdot\vec{\Pi}) - \mathrm{i}m\beta\gamma_5)\phi(\hat{\Lambda})], \qquad \text{where} \quad \phi(\hat{\Lambda}) = \frac{\pi}{4}\frac{1}{\sqrt{-\hat{\Lambda}+m^2}} \tag{2.15}$$

and in this case the algebra is simpler as the whole Hamiltonian is odd, so that $\mathcal{O} = \hat{H}_{\mathrm{PG}}$.

The difficulties arising in the general interacting case, both for $\hat{H}_{\mathrm{D}}$ and $\hat{H}_{\mathrm{PG}}$, are that the kinetic part does not anti-commute with the even terms qA_0, $\beta(\vec{\Sigma}\cdot\vec{B})$, $\beta\gamma_5(\vec{\alpha}\cdot\vec{E})$ and that the interaction parts contain terms of both even and odd parities. Starting with these premises, in the next section we are going to examine the cases in which explicit results can be reached.

3. Discussion of the results

We begin our report on the cases admitting a complete and closed solution by examining a neutral particle, $q = 0$. The Hamiltonian operators in the two different representations are given by (2.8)–(2.10) and both of them involve even and odd terms in the interaction part. We therefore proceed by separating the electrostatic from the magnetostatic interaction.

(a) Starting from the Hamiltonian $\hat{H}_{\mathrm{D}}$ in (2.8) and (2.9), we first consider $q = 0$ and $\vec{B} = 0$ and we investigate the interaction of the anomalous magnetic moment with the remaining electric field. We have

$$\hat{H}_{\mathrm{D}} = \mathcal{O} + \mathcal{E}, \qquad \mathcal{O} = (\vec{\alpha}\cdot\vec{P}), \qquad \mathcal{E} = \beta m, \qquad \text{where} \quad \vec{P} = \vec{p} - \frac{\mathrm{i}e\mu}{4m}\beta\vec{E}. \tag{3.1}$$

In (3.1) we have denoted by $\mathcal{O}$, $\mathcal{E}$ the odd and the even terms, respectively. We then see that the structure is extremely similar to that of the free particle, but for the translation of the momentum by a factor linear in the electric field. Moreover, since

$$[P_i, P_j] = -\frac{e\mu}{4m}\beta\left(\frac{\partial E_j}{\partial x_i} - \frac{\partial E_i}{\partial x_j}\right)$$

for a conservative field, $\vec{\nabla}\times\vec{E} = 0$, our procedure could be considered a canonical transformation with respect to the free case. In the general case, the FWT will be generated by an exponential $\exp\{\beta\mathcal{O}\varphi\}$, where φ is a parameter to be determined in order to obtain a totally even-transformed Hamiltonian. Observe now that a straightforward calculation gives the form of the even term

$$(\beta\mathcal{O})^2 = -\mathcal{O}^2 = -\left[\vec{p}^{\,2} + \left(\frac{e\mu}{4m}\vec{E}\right)^2 - \frac{e\mu}{4m}\beta(\vec{\nabla}\cdot\vec{E} + \vec{\Sigma}\cdot(\vec{E}\times\vec{p} - \vec{p}\times\vec{E}))\right] \tag{3.2}$$

.

Moreover, by parity properties,

$$[\beta\mathcal{O}, \beta]_+ = [\beta\mathcal{O}, \mathcal{O}]_+ = 0, \qquad [\beta\mathcal{O}, \mathcal{O}^2]_- = 0, \tag{3.3}$$

it is easily seen that

$$(\beta\mathcal{O})^3 = -\beta\mathcal{O}^3, \qquad (\beta\mathcal{O})^4 = \mathcal{O}^4, \qquad (\beta\mathcal{O})^5 = \beta\mathcal{O}^5 \quad \text{and so on.} \tag{3.4}$$

We therefore find

$$\tilde{\tilde{H}}_{\mathrm{D}} = \mathrm{e}^{\beta\mathcal{O}\varphi}[(\vec{\alpha}\cdot\vec{\Pi}) + \beta m]\,\mathrm{e}^{-\beta\mathcal{O}\varphi} = [(\vec{\alpha}\cdot\vec{\Pi}) + \beta m]\,\mathrm{e}^{-2\beta\mathcal{O}\varphi}.$$

Due to (3.3) and (3.4) the exponential is easily calculated and yields

$$\mathrm{e}^{-2\beta\mathcal{O}\varphi} = \cos(2\sqrt{\mathcal{O}^2}\varphi) - \frac{\beta\mathcal{O}}{\sqrt{\mathcal{O}^2}}\sin(2\sqrt{\mathcal{O}^2}\varphi) \tag{3.5}$$

so that, expanding the expression of the transformed Hamiltonian, we find

$$\begin{aligned}\tilde{\tilde{H}}_{\mathrm{D}} = \mathcal{O}&\left[\cos(2\sqrt{\mathcal{O}^2}\varphi) - \frac{m}{\sqrt{\mathcal{O}^2}}\sin(2\sqrt{\mathcal{O}^2}\varphi)\right]\\ &+\beta\left[\sqrt{\mathcal{O}^2}\sin(2\sqrt{\mathcal{O}^2}\varphi) + m\cos(2\sqrt{\mathcal{O}^2}\varphi)\right].\end{aligned}$$

Finally, by choosing

$$\varphi = \frac{1}{2\sqrt{\mathcal{O}^2}}\arctan\left(\frac{\sqrt{\mathcal{O}^2}}{m}\right),$$

we find a completely even-transformed Hamiltonian, whose final form is

$$\begin{aligned}\tilde{\tilde{H}}_{\mathrm{D}} = \beta[\mathcal{O}^2 + m^2]^{1/2} = \beta&\left[\vec{p}^{\,2} + m^2 + \left(\frac{e\mu}{4m}\vec{E}\right)^2\right.\\ &\left.-\frac{e\mu}{4m}\beta((\vec{\nabla}\cdot\vec{E}) + (\vec{\Sigma}\cdot(\vec{E}\times\vec{p} - \vec{p}\times\vec{E})))\right]^{1/2}.\end{aligned} \tag{3.6}$$

A similar result has been obtained in a perturbative framework of the FWT in [35–37].

(b) We next assume $q = 0$ and $\vec{E} = 0$, looking for the magnetostatic interaction of the anomalous moment. The Dirac Hamiltonian is

$$\hat{H}_{\mathrm{D}} = (\vec{\alpha}\cdot\vec{p}) + \beta m - \frac{e\mu}{4m}\beta(\vec{\Sigma}\cdot\vec{B})$$

and in it the interaction term is even. If, however, we consider the same problem in the Pauli–Gursey representation, we have a completely odd Hamiltonian

$$\hat{H}_{\mathrm{PG}} = (\vec{\alpha}\cdot\vec{p}) - \mathrm{i}m\beta\gamma_5 + \frac{\mathrm{i}e\mu}{4m}\beta(\vec{\alpha}\cdot\vec{B}). \tag{3.7}$$

The term of (3.7) representing the interaction with the magnetic field has the same structure of interaction term with the electric field of equation (3.1), $(\mathrm{i}e\mu)/(4m)\beta(\vec{\alpha}\cdot\vec{E}) \to (\mathrm{i}e\mu)/(4m)\beta(\vec{\alpha}\cdot\vec{B})$, so that we expect to find a similar result for the diagonalized Hamiltonian, with the obvious replacement of $\vec{E}$ with $\vec{B}$. From the algebraic point of view an analogous result was obtained in [35] when the contribution of EDM was considered. Since the choice of a particular representation is irrelevant for the exact form of the FWT, it is certainly more convenient to start with $\mathcal{O} = \hat{H}_{\mathrm{PG}}$ given in (3.7). As usual we will consider a similarity transformation generated by $\exp(\beta\mathcal{O}\varphi)$, looking, as we previously did, whether we can also satisfy the further requirements $[\beta, \varphi]_- = [\mathcal{O}, \varphi]_- = 0$: if this is the case, we will be able to give a closed form to the FWT and to the transformed Hamiltonian as we did in the previous paragraph. We will give an *a posteriori* solution to these questions.

In analogy with (3.2) we first calculate the even term

$$(\beta\mathcal{O})^2 = -\mathcal{O}^2 = -\Big[\vec{p}^{\,2} + m^2 + \Big(\frac{e\mu}{4m}\vec{B}\Big)^2 - \frac{e\mu}{2}(\vec{\Sigma}\cdot\vec{B}) - \frac{e\mu}{4m}\beta(\vec{\Sigma}\cdot(\vec{B}\times\vec{p} - \vec{p}\times\vec{B}))\Big] \tag{3.8}$$

and the relations (3.3) hold in this case too. Therefore,

$$\widetilde{\hat{H}}_{\mathrm{PG}} = \mathrm{e}^{\beta\mathcal{O}\varphi}\hat{H}_{\mathrm{PG}}\,\mathrm{e}^{-\beta\mathcal{O}\varphi} = \Big((\vec{\alpha}\cdot\vec{p}) - \mathrm{i}m\beta\gamma_5 + \frac{\mathrm{i}e\mu}{4m}\beta(\vec{\alpha}\cdot\vec{B})\Big)\,\mathrm{e}^{-2\beta\mathcal{O}\varphi}$$

and since $\exp(-2\beta\mathcal{O}\varphi)$ is again given by (3.5), if we choose

$$\varphi = \frac{\pi}{4}\frac{1}{\sqrt{\mathcal{O}^2}} \tag{3.9}$$

we find an explicit form for the FW transformed Hamiltonian, that results in

$$\widetilde{\hat{H}}_{\mathrm{PG}} = \beta\sqrt{\mathcal{O}^2} = \beta\Big[\vec{p}^{\,2} + m^2 + \Big(\frac{e\mu}{4m}\vec{B}\Big)^2 - \frac{e\mu}{2}(\vec{\Sigma}\cdot\vec{B}) - \frac{e\mu}{4m}\beta(\vec{\Sigma}\cdot(\vec{B}\times\vec{p} - \vec{p}\times\vec{B}))\Big]^{1/2}. \tag{3.10}$$

One can then verify that all of our working hypotheses are satisfied. Our result agrees completely with the result given in [32]. We also observe that the energy eigenvalues that can be obtained from (3.10) after some straightforward algebra are coincident with those obtained in [38]: the presence of all the three terms containing the magnetic field $\vec{B}$ in (3.10) is obviously essential for the final result.

(c) We finally turn to the case $q \neq 0$. A closed form for the FW transformed Hamiltonian can be found only when $A_0 = 0$. Since, moreover, we consider a stationary case, $\partial\vec{A}(t)/\partial t = 0$, our assumption corresponds to a vanishing electric field. This is very reasonable from a physical point of view, as a non-vanishing $\vec{E}$ could lead to the pair production phenomenon [39, 40]: indeed it is well known that when $\vec{E} \neq 0$, even for a vanishing magnetic field and an anomalous magnetic moment, the FWT cannot be put in a closed form. The model we now discuss can thus describe a proton in a magnetostatic field. We report here the two representations of the Hamiltonian of the system, namely,

$$\begin{aligned}\hat{H}_{\mathrm{D}} &= \beta(\vec{\gamma}\cdot(\vec{p} - q\vec{A}) + m) - \frac{e\mu}{4m}\beta(\vec{\Sigma}\cdot\vec{B})\\ \hat{H}_{\mathrm{PG}} &= \beta\vec{\gamma}\cdot(\vec{p} - q\vec{A}) - \mathrm{i}m\beta\gamma_5 + \frac{\mathrm{i}e\mu}{4m}(\vec{\gamma}\cdot\vec{B}).\end{aligned} \tag{3.11}$$

As in item (b) the second relation in (3.11) shows that the Hamiltonian in the Pauli–Gursey representation is completely odd (i.e., $\hat{H}_{\mathrm{PG}} = \mathcal{O}$) and will be more conveniently used for the FWT. The observations made in (b) concerning the original Dirac and the Pauli–Gursey transformed Hamiltonian apply in the present case too and we can again establish a relation similar to (3.2) and (3.8), that reads

$$(\beta\mathcal{O})^2 = -\mathcal{O}^2 = -\Big[\vec{\Pi}^{\,2} + m^2 + \Big(\frac{e\mu}{4m}\vec{B}\Big)^2 - \Big(q + \frac{e\mu}{2}\Big)(\vec{\Sigma}\cdot\vec{B}) - \frac{e\mu}{4m}\beta(\vec{\Sigma}\cdot(\vec{B}\times\vec{\Pi} - \vec{\Pi}\times\vec{B}))\Big],$$

where Π^μ is the canonical momentum (2.2). The relations (3.3) and (3.4) hold in this case too, so that we can directly write

$$\widetilde{\widetilde{H}}_{\mathrm{PG}} = \hat{H}_{\mathrm{PG}}\,\mathrm{e}^{-2\beta\mathcal{O}\varphi} \tag{3.12}$$

and with the choice (3.9) for the angle φ we get the final form of the transformed Hamiltonian

$$\begin{aligned}\widetilde{\widetilde{H}}_{\mathrm{PG}} = \beta\sqrt{\mathcal{O}^2} = \beta\Bigg[\vec{\Pi}^2 + m^2 + \left(\frac{e\mu}{4m}\vec{B}\right)^2 - \left(q + \frac{e\mu}{2}\right)(\vec{\Sigma}\cdot\vec{B})\\ - \frac{e\mu}{4m}\beta(\vec{\Sigma}\cdot(\vec{B}\times\vec{\Pi} - \vec{\Pi}\times\vec{B}))\Bigg]^{1/2}.\end{aligned} \tag{3.13}$$

We can conclude that the analysis of the 'pseudoclassical mechanics' is quite useful in the derivation of new results. In the present case, in fact, we have analyzed the possible different representations of the Clifford algebra arising from the quantization of the Grassmann variables and we have shown how to extend in a very simple way the quantum unitary transformation which diagonalizes the Dirac Hamiltonian for a particle with anomalous magnetic moment interacting with a classical stationary non-homogeneous electromagnetic field.

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